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On Nilpotent Derivations of Semiprime Rings

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In this paper we study nilpotent derivations of semiprime rings. An associative derivation $d: R \to R$ is an additive mapping on a ring R satisfying $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. A derivation $d: R \to R$ is called *inner* if $d = \operatorname{ad} x$ for some $x \in R$, where $\operatorname{ad} x(y) = xy - yx$. It is proved that for a semiprime ring R, a nilpotent derivation d (with index of nilpotency depending on characteristic) has an extension to the inner derivation and is induced by a nilpotent element of the endomorphism ring $\operatorname{End}(I_R, I_R)$, where I is an essential ideal of R. This is a generalization of some known results due to Kharchenko, Martindale, Chung, and others. © 1992 Academic Press, Inc.

1. INTRODUCTION

Let R be a semiprime ring and let $\mathscr{F}_R$ denote the set of all two-sided ideals of R with zero annihilator. Equivalently, $\mathscr{F}_R$ consists of those ideals of R which are essential as left (or right) ideals. In the case R prime $\mathscr{F}_R$ is simply the set of all nonzero ideals of R. For an ideal $I \in \mathscr{F}_R$ the ring $E(I) = \operatorname{End}(I_R, I_R)$ of right R-module endomorphisms of I is a semiprime ring and $R \subsetneq E(I)$ via the map $a \to a_l$, where a_l is the left multiplication by a acting on I. Moreover, there is a natural embedding of $E(I)$ into the Martindale ring of right quotients $Q_r(R) = \lim_{J \in \mathscr{F}_R} \operatorname{Hom}(J_R, R_R)$. Each derivation $d: R \to R$ leaving I invariant has a unique extension to a derivation $\bar{d}: E(I) \to E(I)$; where $\bar{d}(f)(x) = d(f(x)) - f(d(x))$ for $f \in E(I)$ and $x \in I$. A derivation d is said to be *E-inner* if there is an ideal $I \in \mathscr{F}_R$ and $f \in E(I)$ such that $\bar{d} = \operatorname{ad} f$. It is clear that every E-inner derivation is X-inner, i.e., inner in the Martindale ring of quotients.

In [9] I. N. Herstein proved that if $(\operatorname{ad} a)^n(x) = 0$ for all x in a simple ring of characteristic zero then there exists a scalar λ such that $(a - \lambda)^{[(n+1)/2]} = 0$. W. S. Martindale and R. Miers [10] generalized this

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result to prime rings of characteristic $>n$. In [7] V. K. Kharchenko obtained an important result that all algebraic derivations of prime rings of characteristic zero are X-inner and then extended it to torsion-free semiprime rings [8]. From Kharchenko's and Martindale and Miers' results it follows that nilpotent derivations of prime rings of characteristic zero are X-inner and induced by nilpotent elements. This result was independently obtained also in [5, 11] and generalized by L. O. Chung [1] to the form: if the index n of nilpotency of d is less than characteristic of R then d is E-inner and induced by a nilpotent element in $E(I)$ for some ideal I of R.

It is the main purpose of this paper to extend these results to semiprime rings with a minimal restriction on characteristic. We give a sufficient condition for a nilpotent derivation to be E-inner and induced by a nilpotent element. In particular, we prove that a nilpotent derivation d on a semiprime ring R has this property if for every nonzero d-invariant ideal I of R of prime characteristic p, p does not divide the index of nilpotency of d on I. This means that a nilpotent derivation of a prime ring R is E-inner and is induced by a nilpotent element if the characteristic of R does not divide the index of nilpotency of d (cf. [1]). The idea and the methods of proof in this note are completely different from those used in cited papers.

Throughout this paper, we assume that R is a semiprime ring and d is a nilpotent derivation on R. An ideal I of R satisfying $d(I) \subseteq I$ is called *d-ideal* and denoted by $I \lhd_d R$. If $0 \neq I \lhd_d R$ then I is semiprime as a ring and $d_{|I}$ is a nilpotent derivation on I. For subsets A, B of R we let $\operatorname{r.ann}_A B = \{r \in A \mid Br = 0\}$ to denote the right annihilator of B in A. Similarly, $\operatorname{l.ann}_A B$ will denote the left annihilator of B in A. The differential polynomial ring over R, denoted by $R[X; d]$, is a free left (and right) R-module with basis $1, X, X^2, \ldots$. The addition in $R[X; d]$ is defined as usual for polynomials, but multiplication is extended from R by the rule $Xr = rX + d(r)$. Clearly the ring R can be treated as a subring of $R[X; d]$ and d as a restriction to R of the inner derivation $\operatorname{ad} X\colon R[X; d] \to R[X; d]$.

2. Main Results

We begin with some elementary properties of E-inner derivations. The following lemma will allow us to reduce the problem to semiprime rings of special type.

LEMMA 1. (i) *Let $I \lhd_d R$ and $I \in \mathscr{F}_R$. If $d_{|I}$ is E-inner, then d is E-inner.*

(ii) *Let $R = \bigoplus_{\omega \in \Omega} I_\omega$, where $I_\omega \lhd_d R$. If $d_\omega = d_{|I_\omega}$ is E-inner for each $\omega \in \Omega$, then d is E-inner.*

(iii) *Let* $\{I_\omega \mid \omega \in \Omega\}$ *be a family of d-ideals of R such that* $\bigoplus_{\omega\in\Omega} I_\omega \in \mathscr{F}_R$. *If* $d_\omega = d_{|I_\omega}$ *is E-inner for each* $\omega \in \Omega$, *then d is E-inner. Moreover, if each* d_ω *is induced by a nilpotent element of nilpotency index* $\leqslant k$, *then d is induced by a nilpotent element of nilpotency index* $\leqslant k$.

Proof. (i) Let $0 \neq J \lhd_d I$ *and* $J \in \mathscr{F}_I$. Suppose that $f \in \operatorname{End}(J_I, J_I)$ and $\overline{d_{|I}} = \operatorname{ad} f$. It is clear that $\bar{J} = IJI$ is a d-ideal of both I and R, and $\bar{J} \in \mathscr{F}_R$. We show that $\bar{f} = f_{|\bar{J}}$ is an R-endomorphism. To this end, let $r \in R$, $a, c \in I$, and $b \in J$. Since $ab \in J$ *and* $cr \in I$ we have

$$\bar{f}(abcr) = f(ab)cr = f(abc)r = \bar{f}(abc)r$$

so for each $x \in \bar{J}$ and $r \in R$ $\bar{f}(xr) = \bar{f}(x)r$. Consider the embedding $R \subsetneq E(\bar{J})$ and the derivation $\operatorname{ad} \bar{f}$ of $E(\bar{J})$. It is clear that $(\operatorname{ad} \bar{f})_{|\bar{J}} = (\operatorname{ad} f)_{|\bar{J}}$, so the derivation $\delta = d - \operatorname{ad} \bar{f}$ is a zero map on $\bar{J}$. For $j \in \bar{J}$ and $r \in R$ we have

$$0 = \delta(jr) = \delta(j)r + j\delta(r) = j\delta(r)$$

so $\bar{J}\delta(r) = 0$ and since $\bar{J} \in \mathscr{F}_R$, $\delta(r) = 0$. Thus $d = (\operatorname{ad} \bar{f})_{|R}$.

(ii) For $\omega \in \Omega$ let $J_\omega \lhd_d I_\omega$ and let $J_\omega \in \mathscr{F}_{I_\omega}$. Suppose that $f_\omega \in E(J_\omega)$ and $\bar{d}_\omega = \operatorname{ad} f_\omega$. It is clear that $\bigoplus_{\omega\in\Omega} J_\omega \in \mathscr{F}_R$ and $f = \prod_{\omega\in\Omega} f_\omega$ is an R-endomorphism of $\bigoplus_{\omega\in\Omega} J_\omega$ satisfying $(\operatorname{ad} f)_{|I_\omega} = (\operatorname{ad} f_\omega)_{|I_\omega} = d_\omega$. Hence $\bar{d} = \operatorname{ad} f$, where $\bar{d}$ is a unique extension of d to $E(\bigoplus_{\omega\in\Omega} J_\omega)$.

(iii) This follows immediately from (i) and (ii). ∎

For a subset A of R, let $n(A)$ denote the index of nilpotency of d on A. Let $m(R)$ denote the smallest integer m with respect to the following property:

$$\forall 0 \neq I \lhd_d R \qquad \operatorname{l.ann}_I(d^m(R)) \neq 0.$$

We say that the ring R is *m-homogeneous* (*n-homogeneous*) if for each nonzero ideal $I \lhd_d R$, $m(I) = m(R)$ (resp. $n(I) = n(R)$). A nonzero ideal $I \lhd_d R$ will be called *m-homogeneous* (*n-homogeneous*) if I is m-homogeneous (resp. n-homogeneous) as a ring. Rings (ideals) which are m- and n-homogeneous will be called *homogeneous rings* (*ideals*).

LEMMA 2. *If I is a nonzero d-ideal of R, then* $m(I) \leqslant m(R)$ *and I contains a nonzero homogeneous d-ideal. Furthermore, if R is homogeneous, so is I.*

Proof. Let $m = m(R)$ and let $0 \neq J \lhd_d I$. Then $\bar{J} = IJI$ is a nonzero d-ideal of both I and R, and $\operatorname{l.ann}_{\bar{J}}(d^m(R)) \neq 0$. This implies that $\operatorname{l.ann}_J(d^m(I)) \neq 0$, so $m \geqslant m(I)$.

Now let $0 \neq I \lhd_d R$. Choose an ideal $I^* \lhd_d R$ such that $I^* \subseteq I$ and

$$m(I^*) = \min\{m(J) \mid 0 \neq J \lhd_d R \;\&\; J \subseteq I\}.$$

We claim that I^* is m-homogeneous. Indeed, let $0 \neq J \lhd_d I^*$. For $\bar{J} = I^* J I^*$ using minimality of $m(I^*)$ and since $0 \neq \bar{J} \lhd_d R$, one obtains

$$m(J) \leqslant m(I^*) \leqslant m(\bar{J}) \leqslant m(J)$$

so $m(J) = m(I^*)$, as claimed. By the foregoing it is clear that every d-ideal of R contained in I^* is m-homogeneous. Choosing a d-ideal $I^{**} \subseteq I^*$ with a minimal value of $n(I^{**})$ one obtaines (using similar arguments as above) that I^{**} is an n-homogeneous ideal of R. Thus the ideal I^{**} is homogeneous.

The fact that every nonzero d-ideal I of a homogeneous ring R is homogeneous follows immediately from the previous considerations. ▮

The following proposition and Lemma 1 will allow us to reduce the problem to homogeneous rings.

PROPOSITION 3. *There exists a family $\{I_\omega \mid \omega \in \Omega\}$ of homogeneous d-ideals of R such that*

(i) $\bigoplus_{\omega \in \Omega} I_\omega \in \mathscr{F}_R$,

(ii) *each I_ω is Z-torsion free or has a prime characteristic.*

Proof. By Lemma 2 we can choose a maximal independent set $\{I_\omega \mid \omega \in \Omega\}$ of homogeneous d-ideals of R. Moreover, since every ideal of positive characteristic contains an ideal of prime characteristic, we can assume that each I_ω is Z-torsion free or has a prime characteristic. The annihilator of a d-ideal is also a d-ideal, so by maximality of $\{I_\omega \mid \omega \in \Omega\}$, by the semiprimeness of R, and by Lemma 2 condition (i) holds. ▮

To obtain the main result of this paper we also need the following

LEMMA 4. *Let R be an m-homogeneous ring and let $m = m(R)$. If A is a right d-ideal of R such that* $\mathrm{r.ann}_R(A) = 0$, *then for every nonzero ideal $I \lhd_d R$, $d^{m-1}(A) I \neq 0$.*

Proof. Suppose $d^{m-1}(A) I = 0$. The semiprimeness of R implies $I d^{m-1}(A) = 0$. Let $0 \neq S \lhd_d I$ and $K = S \cap A$. It is clear that $K \neq 0$ and $d(K) \subseteq K \subseteq I$. Choose an integer $k \geqslant 0$ such that $d^k(K) \neq 0$ and $d^{k+1}(K) = 0$. Again, by the semiprimeness of R one obtains $d^k(K) I \neq 0$. Since $d^k(K) I \subseteq A$,

$$0 = I d^{m-1}(d^k(K) I) = I d^k(K) d^{m-1}(I).$$

Thus $0 \neq I d^k(K) \subseteq \mathrm{l.ann}_S(d^{m-1}(I))$, so $m(I) \leqslant m-1 < m(R)$. This contradiction proves that $d^{m-1}(A) I \neq 0$, which gives the result. ▮

Now we are able to prove the main result of this paper.

THEOREM 5. *Let R be an m-homogeneous ring with no Z-torsion or of a prime characteristic p. If p does not divide $m(R)$, then the derivation d is E-inner. Moreover, there exists an essential ideal $I \lhd_d R$, and an element $x \in E(I)$ such that $d = \operatorname{ad} x$ and $x^{m(R)} = 0$.*

Proof. Without loss of generality, we may assume that R is a ring with 1. For otherwise, R can be adjointed by 1 so that the resulting ring R^1 is semiprime. For example the subring R^1 of $Q_r(R)$ generated by R and unity 1 of $Q_r(R)$ has this property.

Consider the ring $R[X; d]$ and its ideal (X^m) generated by X^m, where $m = m(R)$. Let S denote the factor ring $R[X; d]/(X^m)$. Clearly (X^m) is an $\operatorname{ad} X$-invariant ideal of $R[X; d]$, so $\operatorname{ad} X$ induces an inner derivation $\operatorname{ad} x$ on S, where x denotes the image of X in S. Moreover, $x^m = 0$. We claim that $R \cap (X^m) = 0$. Indeed, each element of (X^m) can be represented as a sum of polynomials of the form $f(X)X^m g(X)$, where $f(X), g(X) \in R[X; d]$. Expressing $f(X)X^m g(X)$ as a polynomial with right-hand coefficients one obtains that its free coefficient is a sum of terms of the form $d^m(a)b$, where $a, b \in R$. In particular, each element of $R \cap (X^m)$ can be written as $\sum_{i=1}^{l} d^m(a_i)b_i$, where $a_i, b_i \in R$. Hence

$$R \cap (X^m) \subseteq d^m(R)R. \tag{1}$$

It is clear that $J = R \cap (X^m)$ is a d-ideal of R. Using (1) one obtains that $\operatorname{l.ann}_J(d^m(R))J = 0$. Now the semiprimeness of R and the definition of $m(R)$ gives $R \cap (X^m) = 0$, as claimed. Thus the ring R can be treated as a subring of S and $d = (\operatorname{ad} x)_{|R}$.

Now we will prove by induction that for $k = 1, 2, ..., m$

$$\forall 0 \neq J \lhd_d R \qquad \operatorname{r.ann}_J(x^k) \neq 0. \tag{2}$$

Since $x^m = 0$, (2) holds for $k = m$. Suppose that $k < m$ and $\operatorname{r.ann}_J(x^{k+1}) \neq 0$ for every $0 \neq J \lhd_d R$. Extending the product rule $rx = xr - d(r)$ one obtains $rx^l = \sum_{i=0}^{l} (-1)^i \binom{l}{i} x^{l-i} d^i(r)$. Thus for $l = m$

$$\sum_{i=1}^{m} (-1)^i \binom{m}{i} x^{m-i} d^i(r) = 0. \tag{3}$$

Let $A = \operatorname{r.ann}_R(x^{k+1})$. Clearly A is a right d-ideal of R, so by (3) for every $a \in A$

$$\sum_{i=m-k}^{m} (-1)^i \binom{m}{i} x^{m-i} d^i(a) = 0.$$

Multiplying the last equality by x^k and x^{k-1} we obtain

$$x^k d^m(a) = 0 \tag{4}$$

$$mx^k d^{m-1}(a) - x^{k-1} d^m(a) = 0. \tag{5}$$

Let $0 \neq J \lhd_d R$. If $d^m(A)J \neq 0$, then by (4) $\text{r.ann}_J(x^k) \neq 0$. If $d^m(A)J = 0$, then by (5) $mx^k d^{m-1}(A)J = 0$. Notice that $\text{r.ann}_R(A) = 0$. Indeed, if $B = \text{r.ann}_R(A)$ then $B \lhd_d R$ (because A is d-invariant) and $(A \cap B)^2 \subseteq AB = 0$. By the semiprimeness of R one gets $A \cap B = 0$. The induction assumption implies that $A \cap B = \text{r.ann}_B(x^{k+1}) \neq 0$ provided $B \neq 0$. Hence $B = 0$. Now the assumption on p and Lemma 4 give $d^{m-1}(A)J \subseteq \text{r.ann}_J(x^k)$, which ends the proof of (2).

Let $I = R \cdot \text{r.ann}_R(x)$. Using (2) for $k = 1$ and by the foregoing one obtains that $I \in \mathscr{F}_R$. Moreover, I is d-invariant and for every $r \in R$ and $a \in \text{r.ann}_R(x)$

$$x(ra) = rxa + d(r)a = d(r)a \in I,$$

so the left multiplication by x (denoted by x_l) belongs to $E(I)$. Now it is clear that the unique extension $\bar{d}$ of d to $E(I)$ is of the form $\text{ad}\, x_l$, which gives the result. ∎

Finally we will describe relations between the index of nilpotency and the number $m(R)$.

LEMMA 6. *Let R be an n-homogeneous ring and let $m = m(R)$, $n = n(R)$. Then*

(i) $[(n+1)/2] \leqslant m \leqslant n$;

(ii) *if A is a left d-ideal of R, then $n(A) \geqslant [(n+1)/2]$.*

Proof. (i) Let $0 \neq I \lhd_d R$ and let k be an integer such that $A = \text{l.ann}(d^k(R)) \neq 0$. For all $a \in A$ and $x, y \in R$ we have $0 = ad^k(d^{k-1}(x)\, y) = ad^{k-1}(x)\, d^k(y)$. Then using this identity and $ad^k(d^{k-2}(x)\, d(y)) = 0$ one gets $ad^{k-2}(x)\, d^{k+1}(y) = 0$. Continuing this process we obtain $axd^{2k-1}(y) = 0$, so $ARd^{2k-1}(R) = 0$. Thus $ARd^{2k-1}(AR) = 0$. Since AR is d-invariant, by the semiprimeness of R, $d^{2k-1}(AR) = 0$. The ring R is n-homogeneous, so $2k-1 \geqslant n$. Thus $m \geqslant [(n+1)/2]$.

(ii) Suppose that $n(A) = k$. For all $a \in A$, $x \in R$ we have $0 = d^k(xd^{k-1}(a)) = d^k(x)\, d^{k-1}(a)$. Applying the same argument as in (i) one gets $d^{2k-1}(R)A = 0$, so by the semiprimeness of R, $d^{2k-1}(AR) = 0$ and $k \geqslant [(n+1)/2]$. ∎

A well-known result of L. O. Chung and J. Luh [4] states that the index of nilpotency of a nilpotent derivation on a semiprime ring of characteristic 2 is a power of 2. Moreover, in [2] it was proved that if $\text{char}\, R = p > 2$, then $n = n(R)$ is of the form

$$n = a_s p^s + a_{s+1} p^{s+1} + \cdots + a_l p^l,$$

where $0 \leqslant s \leqslant l$, a_i are nonnegative integers less than p, a_s, is odd, and $a_{s+1}, ..., a_l$ are even. In the case when char $R = p > 2$ and p does not divide $n(R)$, the number $m(R)$ is uniquely determined by $n(R)$.

PROPOSITION 7. *Let R be an n-homogeneous ring with no Z-torsion or of a prime characteristic $p > 2$ and let p not divide $n = n(R)$. Then $m(R) = [(n+1)/2]$, p does not divide $m(R)$, and the ring R is m-homogeneous.*

Proof. Suppose that $m(R) > n_0 = [(n+1)/2]$. Let $0 \neq I \lhd_d R$ let $A = \text{l.ann}_I(d^m(R))$, where $m = m(R)$. By Lemma 6, there exists an integer $t \geqslant 0$ such that $n(A) = n_0 + t$. Moreover, let $m = n_0 + s$ with $s > 0$. For all $a \in A$ and $x \in R$ we obtain

$$0 = d^n(d^t(a)\, d^{s-1}(x)) = \sum_{j=0}^{n} \binom{n}{j} d^{n-j+t}(a)\, d^{j+s-1}(x)$$
$$= \binom{n}{n_0} d^{n_0+t-1}(a)\, d^{n_0+s-1}(x) = \binom{n}{n_0} d^{n(A)-1}(a)\, d^{m-1}(x),$$

so $\binom{n}{n_0} d^{n(A)-1}(A)\, d^{m-1}(R) = 0$.

Now we will use the following well-known result on binomial coefficients (see [6]): if $a = \sum_{i=0}^{l} a_i p^i$ and $b = \sum_{i=0}^{l} b_i p^i$, with $0 \leqslant a_i, b_i < p$ for all i, then $\binom{a}{b} \equiv \prod_{i=0}^{l} \binom{a_i}{b_i} (\text{mod } p)$. Since $n_0 = (n+1)/2 = (a_0+1)/2\, p + \cdots + (a_l/2)\, p^l$ we obtain

$$\binom{n}{n_0} \equiv \binom{a_0}{(a_0+1)/2} \prod_{i=1}^{l} \binom{a_i}{a_i/2} \qquad (\text{mod } p).$$

Clearly the least term is not congruent to zero modulo p, so $\binom{n}{n_0} \not\equiv 0$ (mod p). Thus $0 \neq d^{n(A)-1}(A) \subseteq \text{l.ann}_I(d^{m-1}(R))$. The last inclusion holds for all nonzero d-ideals I, so $m(R) \leqslant m-1$. This contradiction proves that $m(R) = [(n+1)/2]$. The fact that R is m-homogeneous follows immediately from Lemma 6(i). ▌

Combining Proposition 7, Theorem 5, Lemma 1, and Proposition 3 we obtain the following common generalization of some results from [1, 7, 8, 10, 11].

COROLLARY 8. *If d is a nilpotent derivation of a semiprime ring R and if for every d-ideal J of R of a prime characteristic p, p does not divide $n(J)$, then d is E-inner and it is induced by a nilpotent element in $E(I)$ for some $I \in \mathscr{F}_R$.*

3. REMARKS

1. The nilpotent element $x \in E(I) \subseteq Q_r(R)$ in Theorem 5 is uniquely determined by d. Indeed, if $d = \operatorname{ad} x = \operatorname{ad} y$, where $y \in Q_r(R)$ and y is nilpotent, then $x - y = c$ for some element c in the extended centroid C of R. It is clear that $xy = yx$, so $c^k = (x - y)^k = 0$ for a suitable integer k. Since the ring $Q_r(R)$ is semiprime, we have $c = 0$, so $x = y$.

2. Let R be a prime ring and let L be a nonzero left ideal of R. Then for each nonzero ideal I of R $0 \neq IL \subseteq I \cap L$. This implies that $m(R) = \min\{k \mid \operatorname{l.ann}_R(d^k(R)) \neq 0\}$.

3. Every prime ring is homogeneous. In fact, let $0 \neq I \lhd_d R$, $m = m(I)$, and let $L = \operatorname{l.ann}_R(d^m(I))$. By Remark 2 $Ld^{m-1}(I) \neq 0$. Since $d^{m-1}(I)R \subseteq I$, we have

$$0 = Ld^m(d^{m-1}(I)R) = Ld^{m-1}(I)\, d^m(R)$$

so $0 \neq Ld^{m-1}(I) \subseteq \operatorname{l.ann}_R(d^m(R))$. Therefore $m(I) \geqslant m(R)$ and by Lemma 2 one obtains $m(I) = m(R)$. The fact that $n(I) = n(R)$ follows from the well-known result due to L. O. Chung and J. Luh [3].

4. In Proposition 7 we have shown that in n-homogeneous rings of characteristic not dividing $n(R)$ the number $m(R)$ is uniquely determined by $n(R)$. The same is not true when $\operatorname{char} R$ divides $n(R)$. It can be illustrated as follows. Let $p > 2$ be a prime number and let m be an integer such that $(p+1)/2 \leqslant m < p$. Consider a prime ring R of characteristic p containing a nilpotent element x of nilpotency index m and an inner derivation $\operatorname{ad} x$ of R. (For instance, the full $p \times p$ matrix algebra $M_p(F)$ over a field F of characteristic p realizes this situation.) We have $(\operatorname{ad} x)^p = \operatorname{ad} x^p = 0$ and

$$\begin{aligned}(\operatorname{ad} x)^{p-1}(r) &= \sum_{i=0}^{p-1} (-1)^i \binom{p-1}{i} x^i r x^{p-1-i} \\ &= \sum_{i=p-m}^{m-1} (-1)^i \binom{p-1}{i} x^i r x^{p-1-i}\end{aligned}$$

for all $r \in R$. Hence

$$x^{2m-p-1}(\operatorname{ad} x)^{p-1}(r) = (-1)^{p-m} \binom{p-1}{p-m} x^{m-1} r x^{m-1} \neq 0,$$

for some $r \in R$ because R is prime. Thus $(\operatorname{ad} x)^{p-1} \neq 0$ and $n(R) = p$. Moreover, for all $r \in R$,

$$x^{m-1}(\operatorname{ad} x)^m(r) = x^{m-1} \sum_{i=1}^{m-1} (-1)^i \binom{m}{i} x^i r x^{m-i} = 0,$$

so $0 \neq Rx^{m-1} \subseteq \text{l.ann}_R((\text{ad}\, x)^m(R))$. By Remark 2 we obtain that $m(R) \leqslant m$. Now Theorem 5 and Remark 1 imply immediately that $m = m(R)$.

5. Let F be a field of a characteristic $p > 0$ and let $d = d/dx$ be the standard derivation of a polynomial ring $F[x]$. Consider the differential polynomial ring $R = F[x][y; d]$ and its inner derivation $\delta = \text{ad}\, x$. Since $d(x^p) = 0$, the element x^p belongs to the center of R. Thus $\delta^p = (\text{ad}\, x)^p = \text{ad}\, x^p = 0$. On the other hand the symmetric Martindale ring of quotients $Q_s(R)$ is a domain (see [12, Lemma 10.7]). Hence δ is an inner, nilpotent derivation not induced by a nilpotent element of $Q_s(R)$. Notice that $n(R) = m(R) = p$.

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